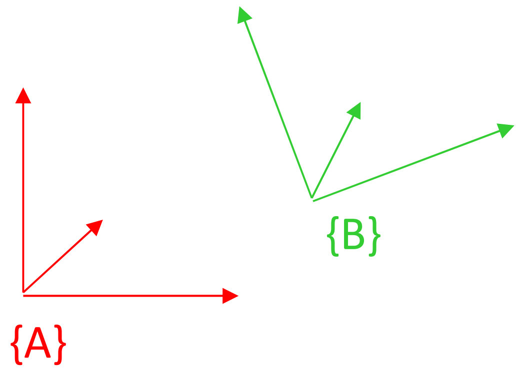
3D-3D Coordinate Transforms

An excellent reference is the book “Introduction to Robotics” by John Craig

3D Coordinate Systems

- Coordinate frames
  - Denote as {A}, {B}, etc
  - Examples: camera, world, model

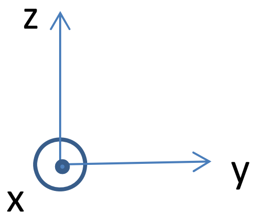


- The pose of {B} with respect to {A} is described by
  - Translation vector **t**
  - Rotation matrix **R**
- Rotation is a 3x3 matrix
  - It represents 3 angles

$$\mathbf{R} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$$

Rotations in 3D

- Rotation about the X axis

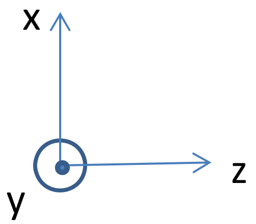


⊙ Points toward me

⊗ Points away from me

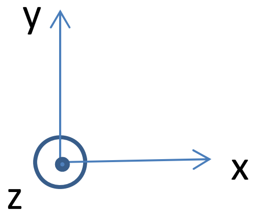
$$\begin{pmatrix} {}^{B}x \\ {}^{B}y \\ {}^{B}z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_X & -\sin\theta_X \\ 0 & \sin\theta_X & \cos\theta_X \end{pmatrix} \begin{pmatrix} {}^{A}x \\ {}^{A}y \\ {}^{A}z \end{pmatrix}$$

- Rotation about the Y axis



$$\begin{pmatrix} {}^{B}x \\ {}^{B}y \\ {}^{B}z \end{pmatrix} = \begin{pmatrix} \cos\theta_Y & 0 & \sin\theta_Y \\ 0 & 1 & 0 \\ -\sin\theta_Y & 0 & \cos\theta_Y \end{pmatrix} \begin{pmatrix} {}^{A}x \\ {}^{A}y \\ {}^{A}z \end{pmatrix}$$

- Rotation about the Z axis



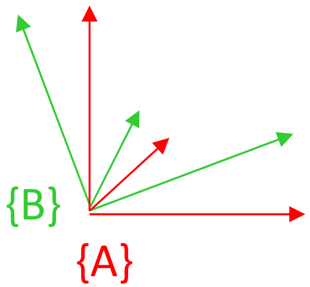
$$\begin{pmatrix} {}^{B}x \\ {}^{B}y \\ {}^{B}z \end{pmatrix} = \begin{pmatrix} \cos\theta_Z & -\sin\theta_Z & 0 \\ \sin\theta_Z & \cos\theta_Z & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} {}^{A}x \\ {}^{A}y \\ {}^{A}z \end{pmatrix}$$

3D Rotation Matrix

- We can concatenate the 3 rotations in any order to yield a single 3x3 rotation matrix
  - e.g., $\mathbf{R} = \mathbf{R}_Z \mathbf{R}_Y \mathbf{R}_X$

$$ {}^B_A\mathbf{R} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} $$

- This is a rotational transformation of frame A to frame B
  - I'll use the leading subscript to indicate "from"
  - I'll use the leading superscript to indicate "to"



- We can rotate a vector in frame A to obtain its representation in frame B

$$ {}^B\mathbf{v} = {}^B_A\mathbf{R}\ {}^A\mathbf{v} $$

- Note: as in 2D, rotation matrices are orthonormal so the inverse of a rotation matrix is just its transpose

$$ \left({}^B_A\mathbf{R}\right)^{-1} = \left({}^B_A\mathbf{R}\right)^T = {}^A_B\mathbf{R} $$

3D Rotation Matrix

- The elements of **R** are direction cosines (the projections of unit vectors from one frame onto the unit vectors of the other frame)

$$
{}^B_A\mathbf{R} = \begin{pmatrix} \hat{\mathbf{x}}_A \cdot \hat{\mathbf{x}}_B & \hat{\mathbf{y}}_A \cdot \hat{\mathbf{x}}_B & \hat{\mathbf{z}}_A \cdot \hat{\mathbf{x}}_B \\ \hat{\mathbf{x}}_A \cdot \hat{\mathbf{y}}_B & \hat{\mathbf{y}}_A \cdot \hat{\mathbf{y}}_B & \hat{\mathbf{z}}_A \cdot \hat{\mathbf{y}}_B \\ \hat{\mathbf{x}}_A \cdot \hat{\mathbf{z}}_B & \hat{\mathbf{y}}_A \cdot \hat{\mathbf{z}}_B & \hat{\mathbf{z}}_A \cdot \hat{\mathbf{z}}_B \end{pmatrix}
$$

- To see this, apply **R** to a unit vector

$$
{}^B_A R \, {}^A\hat{x}_A = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} r_{11} \\ r_{21} \\ r_{31} \end{pmatrix} = {}^B\hat{x}_A
$$

- So the columns of **R** are the unit vectors of A, expressed in the B frame

$$
{}^B_A\mathbf{R} = \left(\begin{pmatrix} \\ {}^B\hat{\mathbf{x}}_A \\ \\ \end{pmatrix} \begin{pmatrix} \\ {}^B\hat{\mathbf{y}}_A \\ \\ \end{pmatrix} \begin{pmatrix} \\ {}^B\hat{\mathbf{z}}_A \\ \\ \end{pmatrix} \right)
$$

- And the rows of **R** are the unit vectors of {B} expressed in {A}

$$
{}^B_A\mathbf{R} = \begin{pmatrix} \left(\quad {}^A\hat{\mathbf{x}}_B{}^T \quad \right) \\ \left(\quad {}^A\hat{\mathbf{y}}_B{}^T \quad \right) \\ \left(\quad {}^A\hat{\mathbf{z}}_B{}^T \quad \right) \end{pmatrix}
$$

Matlab: Creating a Rotation Matrix

```
ax = 0.1;   ay = -0.2;  az = 0.3;   % radians
Rx = [ 1 0 0; 0 cos(ax) -sin(ax); 0 sin(ax) cos(ax)];
Ry = [ cos(ay) 0 sin(ay); 0 1 0; -sin(ay) 0 cos(ay)];
Rz = [ cos(az) -sin(az) 0; sin(az) cos(az) 0; 0 0 1];

R = Rz * Ry * Rx
```

```
R = Rx * Ry * Rz
```

Matlab: Creating a Rotation Matrix

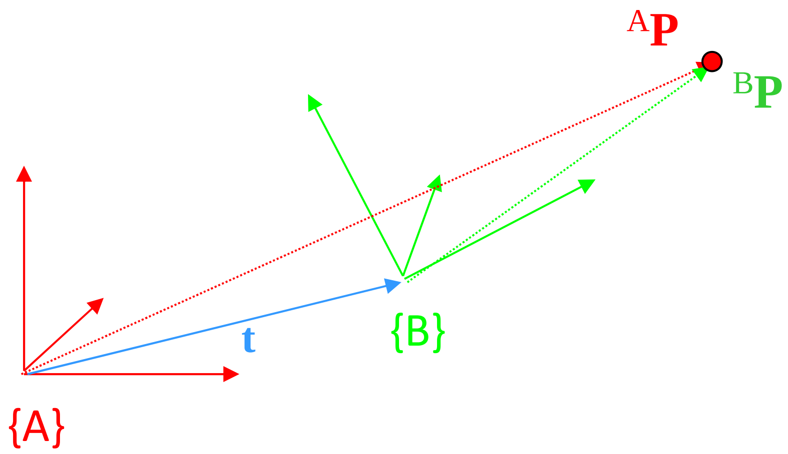
```
ax = 0.1;   ay = -0.2;  az = 0.3;   % radians
Rx = [ 1 0 0; 0 cos(ax) -sin(ax); 0 sin(ax) cos(ax)];
Ry = [ cos(ay) 0 sin(ay); 0 1 0; -sin(ay) 0 cos(ay)];
Rz = [ cos(az) -sin(az) 0; sin(az) cos(az) 0; 0 0 1];

R = Rz * Ry * Rx

R =
    0.9363   -0.3130   -0.1593
    0.2896    0.9447   -0.1538
    0.1987    0.0978    0.9752

R = Rx * Ry * Rz

R =
    0.9363   -0.2896   -0.1987
    0.2751    0.9564   -0.0978
    0.2184    0.0370    0.9752
```

Rotations are not the same

Transforming a Point

- We can use **R,t** to transform a point from coordinate frame {B} to frame {A}

$$ {}^{A}\mathbf{P} = {}^{A}_{B}\mathbf{R}\ {}^{B}\mathbf{P} + \mathbf{t} $$



- Where
  - $^{A}\mathbf{P}$ is the representation of **P** in frame {A}
  - $^{B}\mathbf{P}$ is the representation of **P** in frame {B}

- Note

$$ \mathbf{t} \text{ is the translation of B's origin in the A frame, } {}^{A}\mathbf{t}_{Borg} $$

Homogeneous Coordinates

- We can represent the transformation with a single matrix multiplication if we write **P** in homogeneous coordinates
  - This simply means to append a 1 as a 4th element
  - If the 4th element becomes ≠ 1, we divide through by it

The leading superscript indicates what coordinate frame the point is represented in

$$\longrightarrow {}^{\textcircled{A}}\mathbf{P} = \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} = \begin{pmatrix} sX \\ sY \\ sZ \\ s \end{pmatrix}$$

- Then

$${}^{B}\mathbf{P} = \mathbf{H}\ {}^{A}\mathbf{P}, \quad \text{where } \mathbf{H} = \begin{pmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

General Rigid Transformation

- A general rigid transformation is a rotation followed by a translation

$$ {}^{B}\mathbf{P} = {}^{B}_{A}\mathbf{R}\, {}^{A}\mathbf{P} + {}^{B}\mathbf{t}_{Aorg} $$

- Can be represented by a single 4x4 homogeneous transformation matrix

$$ {}^{B}_{A}\mathbf{H} = \begin{pmatrix} r_{11} & r_{12} & r_{13} & x_0 \\ r_{21} & r_{22} & r_{23} & y_0 \\ r_{31} & r_{32} & r_{33} & z_0 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

- A note on notation:

$$ {}^{A}\mathbf{P} = {}^{A}_{\textcircled{B}}\mathbf{H}\, {}^{\textcircled{B}}\mathbf{P} $$

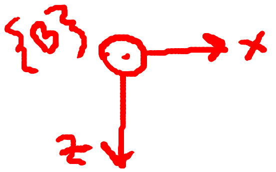
Cancel leading subscript with trailing superscript

$$ {}^{B}\mathbf{P} = {}^{B}_{A}\mathbf{H}\, {}^{A}\mathbf{P} = \begin{pmatrix} r_{11}x + r_{12}y + r_{13}z + x_0 \\ r_{21}x + r_{22}y + r_{23}z + y_0 \\ r_{31}x + r_{32}y + r_{33}z + z_0 \\ 1 \end{pmatrix} $$

Example

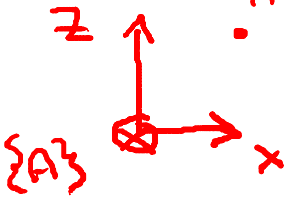
- In coordinate frame A, point P is (1,0,1)
- Frame B is located at (0,0,10) and is rotated 180 degrees about the x axis with respect to frame A
- What is point P in frame B?

WANT ${}^B_A H \rightarrow {}^B P = {}^B_A H \; {}^A P$



$${}^B t_{AORG} = \begin{pmatrix} 0 \\ 0 \\ 10 \end{pmatrix}$$

$${}^B_A H = \left[\begin{array}{c|c} {}^B_A R & {}^B t_{AORG} \\ \hline 0\;0\;0 & 1 \end{array}\right]$$



$$\cdot\, {}^A P = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$${}^B_A R = R_x(180^\circ) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos & -\sin \\ 0 & \sin & \cos \end{bmatrix}$$

$$= \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \\ \begin{array}{ccc} {}^B\hat{x}_A & \quad & {}^B\hat{z}_A \end{array} \end{array}$$

$$ {}^{B}P = {}^{B}_{A}H \, {}^{A}P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 10 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 9 \\ 1 \end{bmatrix} $$

Matlab: Transforming a point

```
% Construct 4x4 transformation matrix to transform A to B
R_A_B = [1 0 0; 0 -1 0; 0 0 -1]   % 3x3 rotation matrix
tAorg_B = [0; 0; 10]   % translation (origin of A in B)

H_A_B = [ R_A_B  tAorg_B;  % H_A_B means transform A to B
          0   0   0   1 ]

P_A = [1; 0; 1; 1]    % A point in the A frame
P_B = H_A_B * P_A      % Convert to B frame
```

Inverse Transformations

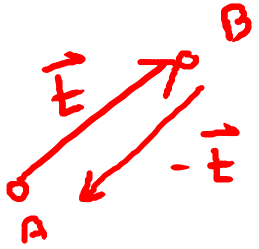
- The matrix inverse is the inverse transformation

$$ {}^{A}_{B}\mathbf{H} = \left({}^{B}_{A}\mathbf{H} \right)^{-1} $$

- Note – unlike rotation matrices, the inverse of a full 4x4 homogeneous transformation matrix is not the transpose

$$ {}^{A}_{B}\mathbf{H} \neq \left({}^{B}_{A}\mathbf{H} \right)^{T} $$

- What is the transformation inverse?



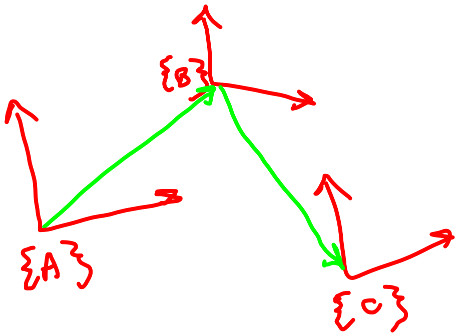
$$ {}^{A}_{B}H = \left[\begin{array}{c|c} {}^{A}_{B}R & {}^{A}t_{BORG} \\ \hline 0 & 1 \end{array} \right] = \left[\begin{array}{c|c} {}^{B}_{A}R^{T} & {}^{A}_{B}R\left(-{}^{B}t_{AORG}\right) \\ \hline 0 & 1 \end{array} \right] $$

Transformations

- Can think of a transformation as:
  - A description of frame {A} relative to frame {B}
  - A transform mapping a point in the {A} frame to its representation in the {B} frame

- Can concatenate transformations together
  - Leading subscripts cancel trailing superscripts

$$ {}^{C}_{A}\mathbf{H} = {}^{C}_{B}\mathbf{H} \; {}^{B}_{A}\mathbf{H} \qquad\qquad {}^{D}_{A}\mathbf{H} = {}^{D}_{C}\mathbf{H} \; {}^{C}_{B}\mathbf{H} \; {}^{B}_{A}\mathbf{H}, \quad \text{etc} $$



Order of Rotations

- XYZ fixed angles
  - Start with {B} coincident with {A}. First rotate {B} about $\mathbf{x}_A$ by angle θ_X, then rotate it about $\mathbf{y}_A$ by θ_Y, then rotate about $\mathbf{z}_A$ by θ_Z.
  - Each rotation takes place relative to the fixed frame {A}
- The order matters
  - Matrices are multiplied in the order Rz Ry Rx
  - Rz Ry Rx order not same as Rx Ry Rz, etc

$$
{}^{A}_{B}R_{XYZ}(\theta_X,\theta_Y,\theta_Z) = R_Z(\theta_Z)\,R_Y(\theta_Y)\,R_X(\theta_X)
$$

$$
= \begin{pmatrix} cz & -sz & 0 \\ sz & cz & 0 \\ 0 & 0 & 1 \end{pmatrix}
\begin{pmatrix} cy & 0 & sy \\ 0 & 1 & 0 \\ -sy & 0 & cy \end{pmatrix}
\begin{pmatrix} 1 & 0 & 0 \\ 0 & cx & -sx \\ 0 & sx & cx \end{pmatrix}
$$

where

$cx = \cos(\theta_X)$, $sy = \sin(\theta_Y)$, etc

$$
{}^{A}_{B}R_{XYZ}(\theta_X,\theta_Y,\theta_Z) = \begin{pmatrix} cz\,cy & cz\,sy\,sx - szcx & cz\,sy\,cx + sz\,sx \\ sz\,cy & sz\,sy\,sx + czcx & sz\,sy\,cx - cz\,sx \\ -sy & cy\,sx & cy\,cx \end{pmatrix}
$$

$$
{}^{A}_{B}R_{XYZ}(\theta_X,\theta_Y,\theta_Z) \neq {}^{A}_{B}R_{ZYX}(\theta_X,\theta_Y,\theta_Z)
$$

Small Angle Approximation

- If rotation angles are small
  - Eg., object is rotating slowly in a video sequence
  - Or we are looking at the effect of small angle perturbations on the rotation
- Then rotation matrix simplifies

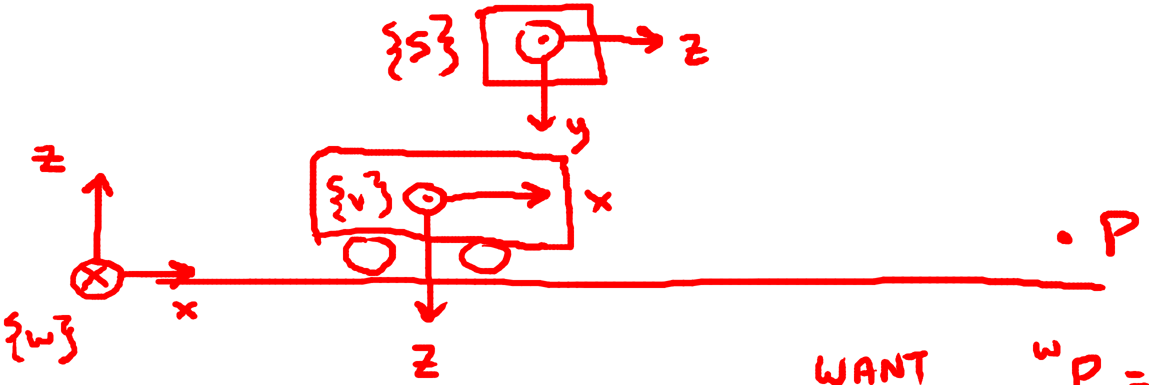
$$ {}_B^A R_{XYZ}(\theta_X,\theta_Y,\theta_Z) = \begin{pmatrix} \cos\theta_Z \cos\theta_Y & \cos\theta_Z \sin\theta_Y \sin\theta_X - \sin\theta_Z \cos\theta_X & \cos\theta_Z \sin\theta_Y \cos\theta_X + \sin\theta_Z \sin\theta_X \\ \sin\theta_Z z \cos\theta_Y & \sin\theta_Z z \sin\theta_Y \sin\theta_X + \cos\theta_Z \cos\theta_X & \sin\theta_Z \sin\theta_Y \cos\theta_X - \cos\theta_Z \sin\theta_X \\ -\sin\theta_Y & \cos\theta_Y \sin\theta_X & \cos\theta_Y \cos\theta_X \end{pmatrix} $$

- Let cos θ ≈ 1, sin $\theta \approx \theta$ for small θ

$$ {}_B^A R_{XYZ}(\theta_X,\theta_Y,\theta_Z) \approx \begin{pmatrix} 1 & -\theta_Z & \theta_Y \\ \theta_Z & 1 & -\theta_X \\ -\theta_Y & \theta_X & 1 \end{pmatrix} $$

Example

- A robot vehicle has a range sensor, which observes a point P. Where is P in world coordinates?



LET $^{s}P = [0, 4, 10]^T$

$^{w}t_{VORG} = [5, 0, 1]^T$

$^{v}t_{SORG} = [1, 0, -2]^T$

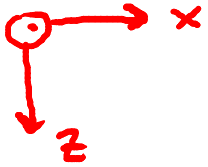
WANT $^{w}P = {}^{w}_{v}H \; {}^{v}_{s}H \; {}^{s}P$

VEH TO WORLD:

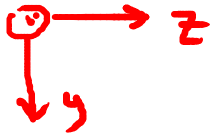
$$^{w}_{v}R = R_x(180^\circ) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$^{w}_{v}H = \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\{V\}$

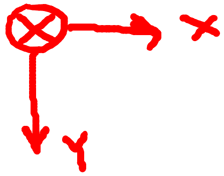


$\{S\}$

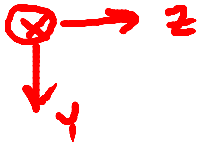


WANT ${}^{V}_{S}R$:

① ROTATE ABOUT ${}^{V}\hat{x}$ (90°)



② ROTATE ABOUT ${}^{V}\hat{z}$ (90°)



${}^{V}\hat{x}_{S}$

$$ {}^{V}_{S}R = R_z(90^\circ)R_y(0^\circ)R_x(90^\circ) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} $$

$$ {}^{V}_{S}H = \left[\begin{array}{c|c} {}^{V}_{S}R & \begin{matrix} 1 \\ 0 \\ -2 \end{matrix} \\ \hline 0 & 1 \end{array} \right] $$

```
clear all
close all


H_V_W = [ 1  0  0   5;
          0 -1  0   0;
          0  0 -1   1;
          0  0  0   1]

H_S_V = [ 0  0  1   1;
          1  0  0   0;
          0  1  0  -2;
          0  0  0   1]

P_S = [ 0; 4; 10;  1]

P_W = H_V_W * H_S_V * P_S
```

WE GET

$$\rightarrow \quad {}^{w}P = \begin{pmatrix} 16 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

Possible Combinations for Rotations

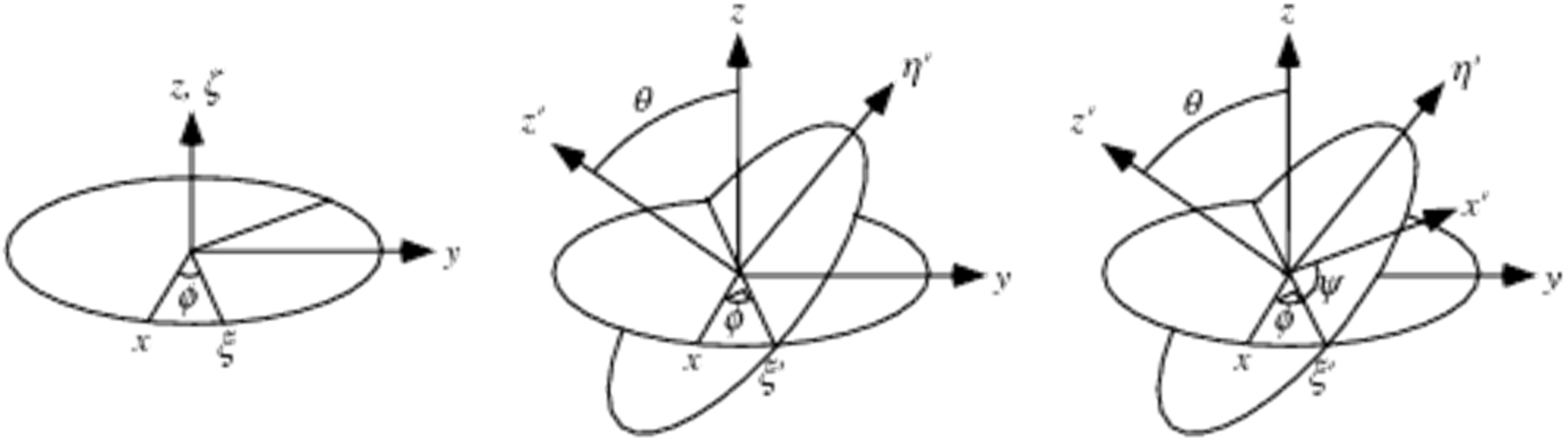
- There are 12 possible combinations for rotations about the fixed axes:
  - $R_X R_Y R_Z$
  - $R_X R_Z R_Y$
  - $R_Y R_X R_Z$
  - $R_Y R_Z R_X$
  - $R_Z R_X R_Y$
  - $R_Z R_Y R_X$
  - $R_X R_Y R_X$
  - $R_X R_Z R_X$
  - $R_Y R_X R_Y$
  - $R_Y R_Z R_Y$
  - $R_Z R_X R_Z$
  - $R_Z R_Y R_Z$

We will use this convention in this course

- So for a given 3D rotation, the values of the 3 angles depends on the rotation convention you use
- However, a given 3D rotation always has a unique rotation matrix

Euler angles – a different rotation convention

- ZYX Euler angles
  - Start with {B} coincident with {A}. First rotate {B} about $\mathbf{z}_B$ by angle θ_Z, then rotate it about $\mathbf{y}_B$ by θ_Y, then rotate about $\mathbf{x}_B$ by θ_X.
  - Each rotation takes place relative to the moving frame {B}



  - There are 12 angle set conventions for Euler angles

Recovering angles from rotation matrix

- Given a 3x3 rotation matrix, and given a certain angle set convention, you can recover the three angles

$$ {}^{A}_{B}R_{XYZ}(\theta_X, \theta_Y, \theta_Z) = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} $$

- Two solutions exist, but can restrict θ_Y to -90° to +90°
- If $\theta_Y = \pm 90°$, we have a degenerate solution: Let $\theta_Z = 0°$

$$ \begin{aligned} \theta_Y &= \operatorname{atan2}\left(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2}\right) \\ \theta_Z &= \operatorname{atan2}\left(r_{21}/cy, r_{11}/cy\right) \\ \theta_X &= \operatorname{atan2}\left(r_{32}/cy, r_{33}/cy\right) \end{aligned} $$

for XYZ fixed angles

$$ \text{If } \theta_{\mathrm{Y}} = +90: \quad \theta_{\mathrm{X}} = \operatorname{atan} 2\left(r_{12}, r_{22}\right) $$

$$ \text{If } \theta_{\mathrm{Y}} = -90: \quad \theta_{\mathrm{X}} = -\operatorname{atan} 2\left(r_{12}, r_{22}\right) $$

Equivalent Angle-Axis

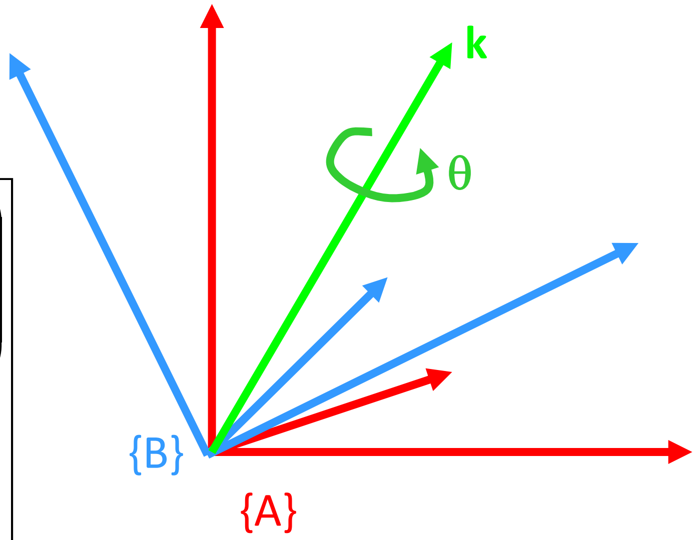
- A general rotation can be expressed as a rotation θ about an axis **k**

$$R_k(\theta) = \begin{pmatrix} k_x k_x v\theta + c\theta & k_x k_y v\theta - k_z s\theta & k_x k_z v\theta + k_y s\theta \\ k_x k_y v\theta + k_z s\theta & k_y k_y v\theta + c\theta & k_y k_z v\theta - k_x s\theta \\ k_x k_z v\theta - k_y s\theta & k_y k_z v\theta + k_x s\theta & k_z k_z v\theta + c\theta \end{pmatrix}$$

where

$$c\theta = \cos\theta,\ s\theta = \sin\theta,\ v\theta = 1 - \cos\theta$$

$$\hat{\mathbf{k}} = (k_x, k_y, k_z)^T$$



- The inverse solution (i.e., given a rotation matrix, find **k** and θ):

- The product of the unit vector **k** and angle θ, $\boldsymbol{\omega} = \theta\mathbf{k} = (\omega_x, \omega_y, \omega_z)$ is a minimal representation for a 3D rotation

$$\theta = \operatorname{acos}\left(\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right)$$

$$\hat{\mathbf{k}} = \frac{1}{2\sin\theta}\begin{pmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{pmatrix}$$

*Note that (-**k**,-θ) is also a solution*